MATH2040 Linear Algebra II

Tutorial 5

October 13, 2016

1 Examples:

Example 1

Let $V = M_{2\times 2}(\mathbb{R}), T : V \to V$ be defined by $T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$ for any $A \in V$, and $W = \{A \in V : A^t = A\}$. Determine whether W is a T-invariant subspace of V.

Solution

By definition, W is a T-invariant subspace of V if $T(w) \in W$ for all $w \in W$.

Note, all matrix in W must be in the form of $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ where $a, b, c \in \mathbb{R}$.

However, $T\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} b & c \\ a & b \end{pmatrix}$, which is not a symmetric matrix and so not in W.

Thus, W is not a T-invariant subspace of V.

Example 2

Let $V = \mathbb{R}^4, T : V \to V$ be defined by $T(a, b, c, d) = (a + b, b - c, a + c, a + d)$, and $z = e_1$.

(a) Find an ordered basis for the T-cyclic subspace of V generated by z.

(b) Let W be the T-cyclic subspace found in (a), find the characteristic polynomial $f_W(t)$ of $T|_W$ and the characteristic polynomial $f(t)$ of T. Show that $f_W(t)$ divides $f(t)$, i.e. $f(t) = f_W(t)g(t)$, where $g(t)$ is some polynomial.

Solution

(a) By theorem, since V is finite-dimensional, so the key step is to find the largest number k such that $\{z, T(z), T^2(z), \ldots, T^{k-1}(z)\}$ is linearly independent, then this set is a basis of the T-cyclic subspace.

Note, $T(z) = (1, 0, 1, 1)$, which is linearly independent of z.

$T^2(z) = (1, -1, 2, 2)$, which is linearly independent of $z, T(z)$.

$T^3(z) = (0, -3, 3, 3) = 3T^2(z) - 3T(z)$, so the largest number k is 3 and $\{z, T(z), T^2(z)\}$ is an ordered basis for the T-cyclic subspace generated by z.

(b) Let α be the ordered basis found in (a) and β be the standard basis of $\mathbb{R}^4$.

Then, $f_W(t) = \det([T|_W]_\alpha - tI_3) = \det \begin{pmatrix} -t & 0 & 0 \\ 1 & -t & -3 \\ 0 & 1 & 3-t \end{pmatrix} = -t^3 + 3t^2 - 3t = t(-t^2 + 3t - 3)$.

And $f(t) = \det([T]_\beta - tI_4) = \det \begin{pmatrix} 1-t & 1 & 0 & 0 \\ 0 & 1-t & -1 & 0 \\ 1 & 0 & 1-t & 0 \\ 1 & 0 & 0 & 1-t \end{pmatrix}$

$$\begin{aligned} &= (1-t)\det \begin{pmatrix} 1-t & -1 & 0 \\ 0 & 1-t & 0 \\ 0 & 0 & 1-t \end{pmatrix} - \det \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1-t & 0 \\ 1 & 0 & 1-t \end{pmatrix} \\ &= (1-t)^4 - (1-t) \\ &= (1-t)[(1-t)^3 - 1] \\ &= (1-t)(-t)[(1-t)^2 + (1-t) + 1] \\ &= (1-t)(t)(-t^2 + 3t - 3). \end{aligned}$$

Example 3

Let A be an $n \times n$ matrix. Prove that

$$\dim(\operatorname{span}(\{I_n, A, A^2, \cdots\})) \leq n.$$

Solution

Suppose the characteristic polynomial of A is $f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, then we have

$$f(A) = (-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n = 0 \tag{1}$$

by Cayley-Hamilton Theorem.

Note, $A^n = (-1)^{n+1}[a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n]$ is a linear combination of $I_n, A, \cdots, A^{n-1}$. After multiplying A on both sides of (1), we have $A^{n+1} = (-1)^{n+1}[a_{n-1}A^n + \cdots + a_1 A^2 + a_0 A]$ is also a linear combination of $I_n, A, \cdots, A^{n-1}$.

Inductively, we have

$$\operatorname{span}\{I_n, A, A^2, \cdots\} = \operatorname{span}\{I_n, A, A^2, \cdots, A^{n-1}\}.$$

Thus, the dimension must not be greater than n.

2 Exercises:

Question 1 (Section 5.4 Q6(d)):

Let $V = M_{2\times 2}(\mathbb{R}), T : V \to V$ be defined by $T(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A$ for any $A \in V$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Find an ordered basis for the T-cyclic subspace of V generated by z.

(b) Let W be the T-cyclic subspace found in (a), find the characteristic polynomial $f_W(t)$ of $T|_W$ and the characteristic polynomial $f(t)$ of T. Show that $f_W(t)$ divides $f(t)$, i.e. $f(t) = f_W(t)g(t)$, where $g(t)$ is some polynomial.

Question 2 (Section 5.4 Q18):

Let A be an $n \times n$ matrix with characteristic polynomial $f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$.

(a) Prove that A is invertible if and only if $a_0 \neq 0$.

(b) Prove that if A is invertible , then $A^{-1} = \dfrac{-1}{a_0}[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n]$.

(c) Use (b) to compute A^{-1} for $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$.

Question 3 (Section 5.4 Q23):

Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Suppose that $v_1, v_2, \ldots, v_k$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $v_1+v_2+\cdots+v_k \in W$, then $v_i \in W$ for all i. (Hint: You may use mathematical induction on k.)

Solution

(Please refer to the practice problem set 5.)